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On the nature and structure of attention in Mathematics Education*

Sobre la naturaleza y la estructura de la atención en la Educación Matemática

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Abstract: It is proposed that not only what people are attending to when speaking or thinking, but how they are attending to it, are both of key significance when people are thinking mathematically, being taught mathematics, reading mathematics education literature, or conducting research in mathematics education. Drawing on the Discipline of Noticing (Mason 2002) and a framework of ways of attending to things (Mason 1986, 2008), the reader is offered experiences though which to try to trap how their attention shifts in both focus and form.

Keywords: Learning of mathematics, Disciplined of noticing, Structure of atention, thinking mathematically.

Resumen: Se propone que no solo a lo que las personas prestan atención cuando hablan o piensan, sino también a cómo lo prestan, son de importancia clave cuando las personas piensan matemáticamente, aprenden matemáticas, leen literatura acerca de educación matemática o realizan investigaciones en educación matemática. Basándose en la disciplina de notar (Mason 2002) y un marco de formas de prestar atención a las cosas (Mason 1986, 2008), se ofrecen al lector experiencias en tratar de atrapar cómo cambia su atención tanto en el enfoque como en la forma.

Palabras clave: Aprendizaje de la matemática, Disciplina de notar, Estructura de la atención, Pensamiento matemático.

Introduction

For some 50 years I have been interested in and concerned about the nature of attention, in relation to mathematics classrooms and mathematical thinking. I came to it by discovering that people in workshops were attending to my accent, which made me wonder what else they were or were not attending to. It struck me that if teacher and learners are attending to different things, and even if they are attending to the same thing(s) but attending to them differently, then communication between them is likely to be at best impoverished and at worst, compromised. That led me to wonder whether the common phenomenon of learners 'not hearing what has been said' could be explained by different foci of attention and different ways of attending.

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Furthermore, it seems to me that teaching is largely about directing and maintaining learner attention, and so Each act of a teacher is an intervention in the attention of learners.

More generally, teaching and learning are different, because Teaching takes place *in* time; learning takes place *over* time.

Teaching involves intervening in learner attention, directing or sustaining it, while learning is a maturation process that takes time (Griffin 1989), and as Gattegno pointed out (1970), mainly takes place during sleep.

Over time what has emerged for me is a five-fold structure of attention in the sense of different ways to attend to the same thing mathematically, based on experiences with J. G Bennett (1993), noting alignment with insights of the van Hieles (van Hiele 1986; Usiskin 1982). There may well be other ways to attend (no claim is made of exclusivity), but the ways outlined here provide insight into what happens when people are engaged in mathematical thinking, in teaching, in mathematics education research, and in professional development.

Methodological Remarks

Because I am interested in the lived experience of mathematical thinking and of supporting the mathematical thinking of others, I take a phenomenological stance. This means that I offer opportunities to experience phenomena, and in a live and physically-present situation there would be discussion about what has been noticed; in a zoom situation I depend mostly on being able to speak to people's experience; and in this written context, I depend totally on being able to resonate with and speak to readers' experience. One consequence is that I do not quote numerous respected sources and parallel writing by colleagues, nor do I smother you with data collected about other people. What you can get from these notes is mainly what you notice happening in your experience as you engage with the tasks, and through resonance with your past experience. This makes it possible for you to imagine yourself acting differently in the future, which is the essence of the Discipline of Noticing (Mason 2002).

Mathematical Thinking

Context 1

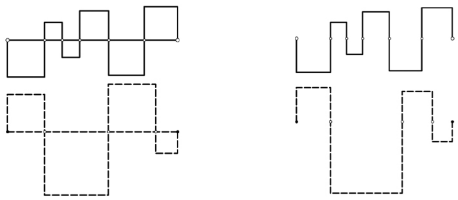
The first task is based on one that appeared recently on twitter, from Prof. Smudge.

Task 1: Prof Smudge's Perimeters

On the left of Figure 1 there are two objects on the left made up of squares, one solid, one dashed. The question is which collection of squares appears to have the larger perimeter?

Figure 1

Squares and Paths



Task 1A: Prof Smudge's Paths

In Figure 1, there are two paths on the right, one solid, one dashed, each made up of three sides of a square. Which path appears to be the longest?

Which actually is the longest?

Conjectures

I suspect that your first impression on catching sight of the figures (before or after reading the task) was a sense of shapes. It is useful to call this *gazing*, or slightly more technically, *holding wholes* (the alliteration in English can be helpful as an aide to memory). It may be only for a few micro-seconds, but in mathematics it can be for extended periods of time as one waits for inspiration.

An early impression might have been to discern various squares, and to have a sense that there are more solid squares than dashed squares, and the dashed squares look bigger; but an inner witness may quickly ask why the question was asked unless there was a trick … so … .

At some point you *discerned details* such as, perhaps, that the starting and ending points are vertically aligned. You may have discerned (noticed) that the paths are made up of segments.

At some point you may have *recognised relationships* between the lengths of some of those segments, as appearing to be equal, and this may have shifted your attention back to the original task, or brought to mind that in the task it says that the paths are made up of edges of squares.

At some point you may have *perceived* the possible relevance of the *property* of squares that all the edges are of equal length. This might have sparked off a different focus when *discerning details* freshly, recognising the square-relationships, and perceiving the edges property as pertinent.

Putting all this together, you now attend to the whole differently, *reasoning on the basis of* (by reference to) agreed properties of squares.

The italicised terms constitute some ways of attending that can be discerned and related, and which will be exploited in what follows.

Holding Wholes (gazing)
Discerning Details
Recognising Relationships (in the situation)
Perceiving Properties (as being instantiated)
Reasoning on the Basis of Agreed Properties

People familiar with van Hiele 'levels' will recognise a strong alignment. What is notably different is that personal observation suggests that attention can flit quickly between different forms, as well as remain in one form for a period of time. The aim is not to classify or label learners, but rather to recognise one's own form and focus of attention, in comparison with what seems to be the case from what learners are saying and doing, in order to inform pedagogic choices. More on this later.

Comment

It is not easy to trap the movement and changes in your attention. Once some relationship is recognised it is difficult to return to a state of non-recognition. Further opportunities to trap changes in the focus and form of attention are available in the next and following tasks.

Task 2: What makes it work?
What is it that makes the two paths forced to have the same lengths?

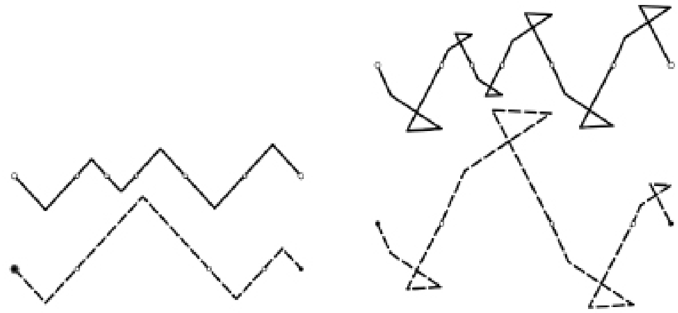
Comment

Brown & Walter (1983) suggest the pedagogic device of inviting people to read out the original task placing stress on a particular word. This seems to raise the question of 'why this word?' or 'what else could this be?', in alignment with the variation principle (Marton 2015, Watson & Mason 2005) which prompts the question of 'what are the dimensions of possible variation in this task?' and 'What is the scope and range of permissible change?' which leaves the equality of the two paths the same.

For example,

Figure 2

Two more complex examples of paths



Task 3: More Paths
In Figure 2, there are two more pairs of complex paths. In each case, which path, the solid or the dashed is the longer?

Comment

Because this is a written exposition, the task for a reader is to make sense of the examples given, whereas in a live workshop, participants would be urged to raise questions as in the previous task, and to explore the dimensions of possible variation and the ranges of permissible change.

Note: the diagrams are generated by the applet Smudge's Perimeters (PMTheta 2021), which is freely available, and permits various perturbations to paths, exploring Tasks 2 and 3.

Comment

It is highly likely, given my experience using this task, that you had moments of gazing, of discerning details (such as the segments making up a similar shape), of recognising relationships between those details (how they contribute to similar copies of a single shape), and perceiving properties (realising that the path length contributed by each component shape is the same multiple of one of the shape's edges), enabling you then to reason on the basis of agreed properties of perimeters of similar shapes.

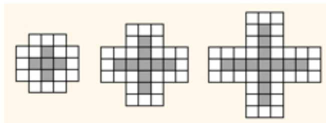
Context 2

In the talk I used a picture from the internet (DandoesMaths website) which may be subject to copyright, so the next task is a more familiar type of task associated with expressing generality as an introduction to algebra.

Task 4: Tiling Surrounds

Here are the first three pictures in an ongoing sequence Figure 3.
Determine a rule for building each picture in the sequence.

Figure 3
Three initial pictures in a sequence



How many white tiles are needed to make the next pattern according to your rule?
How many white tiles are needed to make pattern 10?
Suggest a method to calculate the number of white tiles needed in any pattern in the sequence
Suggest a method to calculate the number of white tiles needed for the *n*th pattern in the sequence

Comment

This task is taken from Amit & Nera (2021). I have added in the instruction to determine a rule for building subsequent pictures which are consistent with the pictures, because no pattern is uniquely specified without some indication of how it is generated (see for example Johnston-Wilder & Mason 2005).

Conjectures

Even if you are familiar with this type of task, it is very likely that there was a moment when you were gazing without really absorbing detail, perhaps with a shift of attention to a sense of familiarity, or even to considering how the task would be formulated in detail. This might have led you to discerning details, either pedagogic or mathematical.

The tasks was used originally to highlight the difference between successively extending pictures and being able to articulate how to draw any particular pattern in the sequence. What is available to notice is the different ways you need to attend, the different type of relationships which then extend to properties, that are required in order to 'see' the pattern either as growing successively, or as something which can be drawn *ab initio*. This is of significant pedagogic importance in the learning trajectory constituting exposure to the elementary ideas of algebra.

Reflection

These two task domains are intended to indicate that attention is important in mathematical thinking, both what is attended to, and how it is attended to.

Mathematics Teaching

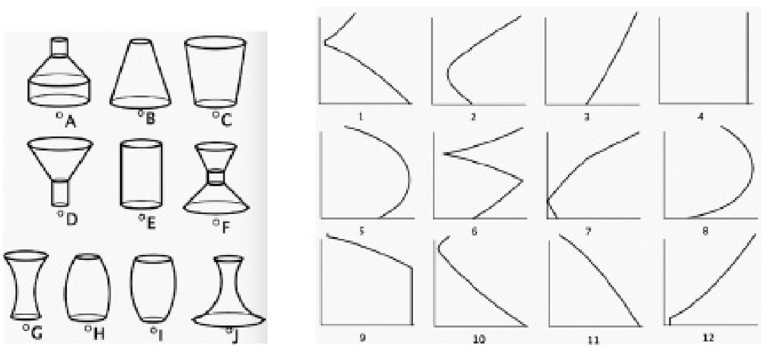
The next task is based on a familiar idea, but oriented towards a difficulty that may be quite widespread … which cannot be announced until after the task has been tackled!

Task 6: Flask Filling

Figure 4 displays 10 flasks (vessels with a circular cross section) which can be filled with liquid. It also displays 12 graphs, 10 of which correspond to the graphing of surface area of the liquid against volume of liquid in the flask as the flask is filled.

Figure 4

Flasks and graphs of surface area against volume as the flasks are filled



The task is to match the graphs to the flasks, and to sketch the two flasks whose graphs are shown but the flasks not depicted.

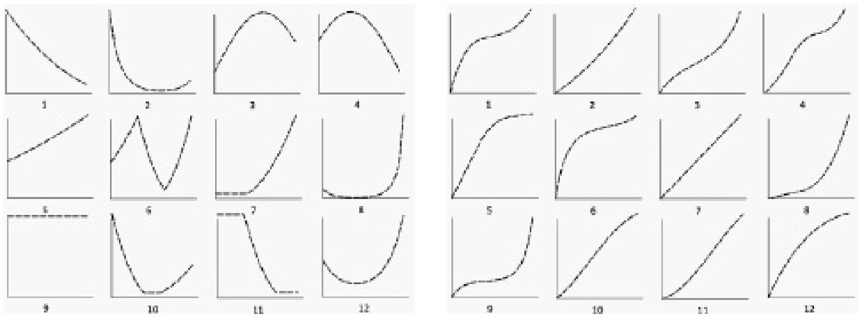
Conjectures

It is highly likely that unless you have worked recently with phase diagrams, you will find this a difficult task. Pedagogically, it may be useful to put learners in a situation in which they find a task very challenging, so that they can experience the need for, and usefulness of specific aspects of mathematical thinking which might be helpful. In this case, asking for, or drawing for themselves, graphs of surface area against height and volume against height, as in Figure 5.

Figure 5

Graphs of surface area and volume against height

The sequences have been deliberately altered so as to continue to present a challenge



As additional support, the graphs of surface area against height and volume against height can be superimposed. All of these variations are available in the applet developed to display these images (Mason 2021a).

Pedagogical Comment

Having tried the phase-diagram task for oneself, and tried to trap how to direct ones attention (focus and form), making a note of the way in which particular flasks proved helpful to get started and from which to migrate to harder examples, it is then possible to make some pedagogical choices about which graphs to start with, so that specific learners can cope but still be challenged. Pedagogical preparation involves considering how you might direct attention when it seems necessary so that learners can make progress and reason things out for themselves. No task is an island complete unto itself, either because of pedagogical choices to be made, or because of opportunities to extend and broaden. For example, how might a flask vary and still have the same graph?

Comment

The intention of the task is to highlight how attention needs to be deliberately controlled in order to identify which graph belongs to which vase. This involves discerning details in the shape of the graph and the shape of the flask, perceiving properties (what happens as the height of liquid rises) and instantiating these to the particular graph and flask. An even harder task is to be asked to draw the graph for yourself for a given flask, or to determine possible shapes for a flask given the graph.

Seeing learning as climbing a staircase (a *staircase* pedagogy) makes the mistake of training learners in dependency on the teacher and thereby even decreasing engagement, instead of challenging learners appropriately, in a context of support for mathematical thinking so as to deal with being stuck (Coles 2021).

Mathematics Education Literature

My next conjecture is that attention plays a key role in the use of technical or theoretical terms in mathematics education literature. Mathematics Education literature abounds with technical terms, introduced by authors and then re-interpreted by colleagues, sometimes beyond all recognition. Here are two examples.

Transposition Didactique

Introduced by Yves Chevellard (1985) and then greatly extended by him:

> The process of didactic transposition refers to the transformations an object or a body of knowledge undergoes from the moment it is produced, put into use, selected, and designed to be taught until it is actually taught in a given educational institution. [Chevellard (1985), quoted in Encyclopaedia of Mathematics Education]

When I first encountered the construct, it spoke immediately to the transformation from an expert encountering a mathematical idea in some context, to a task for learners, intended to offer them the same or similar experience. My version was then

> The transformation of expert awareness into learner behaviour (Mason 2004).
> Not only are these rather different in length and scope, but also in what is being stressed.

Task 7:

What is being attended to in these two definitions? How are they different in how they are being attended to?
To address these questions, try asking yourself ‘what would I have to be attending to, in order to come up with that definition?

Comment

The first offers more specific details of stages of transformation than does the second. The second makes use of further technical terms (awareness, behaviour) which have their own history and multiple interpretations according to author. To express the first version, I think I would have to be stressing a sociological perspective, seeing the process as the result of a system transposing the original into teaching material. To express the second, I know that I was focusing on my direct experience and realising the range of choices I could make during the transposition. The first seems to me to invoke reasoning on the basis of (sociological) properties, while the second seems to invoke a shift from recognising relationships in a particular situation to perceiving a property of textbook construction.

Task 7A:

For either or both of Perimeters and Flasks, what didactic transposition did the core idea undergo between the original and the version offered here?
Consider how adopting the two different perspectives might lead to (slightly?) different realisations and future behaviour in constructing tasks.

I can report that the second expression of the didactic transposition came about from recognising in my own experience many times coming across a problem, concept-instantiation or task (such as Prof Smudge's Paths, and the Flask Task) and having a sudden (if slight) epiphany, a realisation. This then triggered a recognition that the task-situation might be used to provide learners with a useful experience related to or stemming from my own experience. The conversion then takes place into a sequence of instructions to learners. Where I made use of my mathematical powers, it is tempting for the final task to be instructions to learners concerning which acts to carry out, which actions to initiate. Consequently learner experience is likely to be a far cry from my original experience.

As Chevellard indicates, there may be a sequence of 'experts' who make transformations in the task before it actually gets to the learners, and even after the teacher has presented the task, there can be a difference in the task that learners undertake as they interpret the task for themselves!

Didactic Tension

In a session led by Guy Brousseau at ICME 5 in 1984, the notion of a fundamental teacher dilemma was mooted. It resonated with other expressions (Holt 1965 p37-38; Driver 1983 p3, p49) and over the following years it reached articulation in mathematics education in various places:

> Everything he [the teacher] does to make the pupil produce the behaviour he expects tends to deprive the pupil of the conditions necessary for understanding and learning the notion concerned. If the teacher says what he wants, he can no longer obtain it (Brousseau 1984, p. 110).

I rearticulated it several times, using the name *didactic tension*:

> Everything the teacher does to make the pupil produce the behaviour the teacher expects, tends to deprive the pupil of the conditions necessary for producing the behaviour as a by-product of learning; the behaviour sought and the behaviour produced become the focus of attention (Mason 1986, p. 21-2).
> The more explicitly the teacher indicates the behaviour sought, the easier it is for the students to display that behaviour, without generating it through understanding (Mason & Davis 1988).

It was then quoted and built upon:

> … a didactic tension (Mason, 1988) arises in practice, with teachers then stuck in a position of having certain knowledge to inculcate or elicit, while recognizing that such knowledge is individual and can only be shared through listening and negotiation. Jaworski (1994), quoted by Pratt (2003).

Task 8: Didactic Tension
What might you have to attend to, and how, to express the notion of a didactic tension in these different ways.

Conjecture

When appropriating technical terms in mathematics education, it might be useful to seek out alternative expressions, and to ask yourself what the focus and form of attention would have to be to reach each particular articulation. This might then inform how the technical term is used in your own analysis.

Researching

When conducting extra-spective research (studying the behaviour or experience of other people), it can be informative, before employing other distinctions, to ask yourself what it is that you would have to be attending to, and in what way, in order to behave as your subjects behave. In this way you access at least a conjecture as to what the subjects were thinking and doing, or a starting point for modifying that conjecture. In this way you can avoid simply showing that your chosen distinctions can be used as descriptors for your data.

What is Lucy Attending To?

Here is an example of locating possible attention foci and forms.

Task 9: Lucy

Try to catch the shifts in focus and form of your attention as you try to make sense of what Lucy says and does in the following brief account.

The following account is offered in Amit & Neria (2010)

Lucy: I need a sum [of digits] to be something divisible by 9. So we can choose 18. OK, 8 and 2 give 10, and now I break the 8 [the units of 18] into 4, 3 and 1. So 82431 should be divisible by 9.
Interviewer: Can you think of another strategy to find such a number?
Lucy: I could make the sum something else, like 27 or something...

Comment

It is necessary to make a conjecture as to what Lucy is trying to achieve … for example, the construction of numbers divisible by 9. Notice that her use of 8 in two different ways is a potential source of confusion on first reading. My attention was focused on the 8s as I started reading so I had to let go of that fixation in order to see the 8's as 'different' before having any sense of what Lucy was up to.

Conjecture

Before calling upon whatever analytical frameworks have been selected for analysing data, it might be useful to ask the question 'What would I have to be attending to, and how, in order to say (or do)what this subject has said (or done)?'.

A useful way to begin analysis is to ask oneself "What would I have to have been attending to, and how, in order to say or do what this person has said and done?". It seems to me that Lucy has a particular 'method' in mind. She seems to attend to the ten of 18, selecting 8 and 2 as digits to sum to ten. Then she decomposes the eight of 18 as a sum and juxtaposes all these digits to form a large number. So it is likely that her attention is governed or restricted by sequentiality. In this transcript there is no suggestion that she is aware that the digits she selects can be in any order, for example. Her attention is on selecting digits which will have the same sum as the number she started with (18) which, being a multiple of 9, will then be divisible by 9.

Making sense of the transcript word by word is unlikely to be successful … discerning her answer, 82431, and how it is made up of her previous utterances, led me to recognising a relationship which is a single instantiation of a general principle for constructing numbers divisible by 9. But notice how Lucy is bound by starting with a number known to be divisible by 9, and then presumably breaking up the digits in a similar fashion to the 18 example. So she may not have perceived the general property, that the sum of the digits simply have to sum to 9, but rather she is currently locked into her self-derived 'method'.

Conclusion

All experience is determined by the current focus and form of attention, strongly influenced by recent and past enculturation. By asking oneself about any behaviour, whether words or actions,

"what would I have to be attending to, and how, in order to behave in that way?"

is likely to open up pedagogical as well as theoretical possibilities for action. The five forms of attention: holding wholes, discerning details, recognising relationships, perceiving properties, and reasoning on the basis of agreed properties provide a rich starting point for sensitising oneself to the focus and form of others.

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